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MOTION OF POINT VORTICES AS A DYNAMICAL SYSTEM

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ABSTRACT

Motion of point vortices are investigated in terms of a Hamiltonian dynamical system. The integrability of the system is surveyed using the Liouville theorem. As aresult, threevorticesin an unbounded regionare integrable, while the existence of boundaries or external flows diminish the number of the first integrals, and we may expect chaotic behaviors for fewer vortices.

1. INTRODUCTION

The study of vortex motion plays an essential role in understanding the complex but fundamental aspects of fluids, especially the mechanism of turbulence. Among others, the motion of point vortices not only gives crucial insight of vortex motion both qualitatively and quantitatively, but also arouse our interest in its structure as a dynamical system, because of the fact that the motion is derived under the Hamiltonian formalism.

In recent active studies of so-called chaos, number of works have been done on the integrability of the motion of point vortices since Novikov[1)] shed light on the stochasticity of them. Originally, the stochasticity of point vortices was conjectured by Onsager in 1949[2)], but his argument was based on many vortices. Novikov's approach was new because he took notice of a few vortices, which is closely related to the integrability of the Hamiltonian of the motion. Extending Novikov's idea, Aref[3),4)] presented the criterion of the integrability in terms of the competition between the freedom of the system, i.e. the number of vortices, and the number of the first integrals, such as energy, the center of vorticity and the moment of inertia for the unbounded case. Thus when vortices are in an infinite region and there is no external flow, the threshould number of the integrable case should be three according to the Liouville theorem. (see § 2)

While when vortices are in a finite region or in some external flow, fewer than four are expected to be chaotic as predicted by Novikov because the boundaries or the external flows reduce the symmetry of the system, resulting in the loss of some first integrals.

In this article, we first present the Hamiltonian formalism of the motion of point vortices and the first integrals owing to the symmetry of the system, then discuss the case when a boundary or an external flow exist. We also comment on the nonexistence of hidden integrals of the system.

2. FORMULATION

The motion of N point vortices in an unbounded region is governed by the following equations,[5)]

$$\frac{dz_j}{dt} = -\frac{1}{2\pi i}\sum_{k=1}^{N}{}' \frac{\Gamma_k}{\bar{z}_j - \bar{z}_k} \qquad (j=1,2,\ldots N), \tag{2.1}$$

where z_j and Γ_j are the position of j-th vortex in the complex Z-plane and its strength, respectively. The prime on the summation denotes the omission of the singular terms j=k, and the bar on the variables means the complex conjugate.

Equation (2.1) is obtainable by the following Hamiltonian,[6]

$$H = -\frac{1}{4\pi} \sum_{(i \neq j)} \sum \Gamma_i \Gamma_j \log |z_i - z_j| \quad . \tag{2.2}$$

If we define the poisson bracket as

$$\{ f, g \} = \sum_{j=1}^{N} \frac{1}{\Gamma_j} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x_j} \right) , \tag{2.3}$$

where x_j, and y_j are conjugate variables, we can describe the equations of motion as

$$\dot{x}_j = \{ x_j, H \} \quad , \qquad \dot{y}_j = \{ y_j, H \} \quad . \tag{2.4}$$

We should notice that the Poisson bracket is a little different from the usual use by the factor $1/\Gamma_j$. It is because the strengths of vortices can be negative, otherwise we can replace the variables with those factored by $1/\sqrt{|\Gamma_j|}$ so that the Poisson bracket remains the same. For the later use, we also give the expression of the equations of motion in terms of complex variables,

$$\Gamma_j \dot{\bar{z}}_j = 2i \frac{\partial H}{\partial z_j} \quad . \tag{2.5}$$

It is easily verified that the Hamiltonian, eq.(2.2) has the following quantities as the first integrals, i.e., they commute with the Hamiltonian in terms of the Poisson bracket,

1) Moment of Inertia,

$$I = \sum_{j=1}^{N} \Gamma_j |z_j|^2 , \tag{2.6}$$

which is due to the rotational symmetry of the system about the origin,

2) Components of the Center of Vorticity,*

$$G_x = \sum_{j=1}^{N} \Gamma_j x_j , \qquad G_y = \sum_{j=1}^{N} \Gamma_j y_j , \tag{2.7}$$

which is due to the translational symmetry of the system.

The check of the integrability of the Hamiltonian is based on the following powerful theorem by Liouville,[7),8)]

Theorem (Liouville)

> A Hamiltonian in N degrees of freedom can be solved by quadratures if N independent integrals in involution exist.

It is shown that among above first integrals, I and G_x (also I and G_y) are in involution, i.e.

$$\{ I, G_x \} = 0 \qquad (\text{or} \quad \{ I, G_y \} = 0) , \tag{2.8}$$

but G_x and G_y are not;

$$\{ G_x, G_y \} = \sum_{j=1}^{N} \Gamma_j . \tag{2.9}$$

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* This definition is different from the usual one by the factor $1/ \sum_{j=1}^{N}\Gamma_j$

In order that the integrals are to be in involution, we should notice that

$$\{ H, G_X^2 + G_Y^2 \} = 0 \tag{2.10}$$

which leads us to the conclusion that the equation of motion is perfectly integrable for at least three vortices in an unbounded region. Equation (2.9) shows , however, that G_x and G_y will commute if the total strength vanishes, which suggests the possibility of the integrable case of four vortices.[8)]

If there exists a boundary or an external flow, some of the first integrals are lost, and the threshould number of the integrable case becomes lower. For such examples, we will mention in the following sections, the case of two vortices in a semicircle as a boundary, and in a rotating uniform shear as an external flow. With the former, the moment of inertia is no longer conservative because the rotational symmetry is broken, and we may expect a chaotic behavior. While with the latter, the Hamiltonian is decoupled into two parts, each of which depends only on the coordinates of the center of vorticity, and that of the relaive motion, respectively, which implies that this case is integrable.

3. MOTION OF VORTICES IN A SEMICIRCLE

When vortices are in a circular region, we should consider the effects of the mirror images to satisfy the boundary conditions. If we assume the radius of the circle to be unity for the sake of simplicity, the equations of motion of vortices become

$$\frac{dz_j}{dt} = -\frac{1}{2\pi i}\left(\sum_{k=1}^{N} \frac{\Gamma_k}{\bar{z}_j - \bar{z}_k} - \sum_{k=1}^{N} \frac{\bar{z}_k \Gamma_k}{\bar{z}_k \bar{z}_j - 1} \right) \quad (j=1,2,...N) \tag{3.1}$$

whose second term in the right hand side corresponds to the mirror images.[9)-12)]

We should replace the previous Hamiltonian to the new one,

$$H = -\frac{1}{4\pi}\sum_{j=1}^{N} \Gamma_j^2 \log \frac{1}{1 - z_j \bar{z}_j} - \frac{1}{4\pi}\sum_{j=k}\sum \Gamma_j \Gamma_k \log \frac{|z_j - z_k|}{|1 - z_j \bar{z}_k|} \tag{3.2}$$

In this case, only the moment of inertia survives for the first integral.

In order to investigate the motion of two vortices in a semicircle, we locate two vortices in a circle at z_1 and z_2, and their mirror images at $\bar{z}_1$ and $\bar{z}_2$ as shown in Fig.1. Thus this problem constitute the restricted eight body problem. As far as the two vortices in the semicircle are concerned, the moment of inertia is no longer the integral, because the rotational symmetry is lost.

3-1 Simulation[9),10),12)] Hereafter we present the results of the numerical simulation of the motion of two vortices of equal but opposite circulation in a semicircle, denoted by their circulation Γ_1 (=1) and Γ_2 (=-1).

In undertaking the simulation, the first step is to find some particular solutions like, fixed points or periodic orbits.[11)] We notice that two equilibrium configurations exist. One of them is stable, $(r_1, \theta_1) = ((\sqrt{17}-4)^{1/4}, \pi/4)$ and $(r_2, \theta_2) = ((\sqrt{17}-4)^{1/4}, 3\pi/4)$, and the other is unstable, $(r_1, \theta_1) = (0.75264\cdots, \pi/2)$ and $(r_2, \theta_2) = (0.25578\cdots, \pi/2)$. From them, we choose the former and apply perturbations to r_2, i.e. by shifting Γ_2 along the line$\theta = 3\pi/4$, while Γ_1 is left on the original fixed point as the initial positions. (see Fig.2)

As the scheme of the calculation, Runge-Kutta method of the fourth order is adopted, and the time step is adjusted to keep $E = \exp(-8\pi H)$ constant to the eight decimal places. We will show the trajectries of two vortices in the semicircle for some initial positions. Besides, we add the power-spectrum of one variable r_1 in the time series for each cases as a direct evidence of regularity.

As long as the perturbation is small, the trajectires are elliptic and two dominant frequencies are shown in the spectrum. (Fig. 3(a)) With much perturbation, the trajectries begin to exhibit complicated features, though the total shape remains symmetric about the y-axis. We can observe tiny peaks besides the two dominant ones in the spectrum.(Fig. 3(b)) These peaks gradually grow with the perturbation until they suddenly seem continuous and the corresponding trajectries become entangled. (Fig. 3(c) and 3(d))

With further shift of Γ_2 to the origin, however, two regions that have high density appear in the entangled trajectries, and the corresponding spectra grow. (Fig.3(e)) At last, two vortices are separated into two regions, and the mixing is no longer seen, and two distinctive spectra are observed. (Fig.3(f)) We observe that the transition from the scattered state to the confined one occures very sharply by the small change of the initial value of r_2 (about 10^{-3} percent). At the limit, these regions shrink to narrow lines on which vortices move in the same direction with a small oscillation. The outer

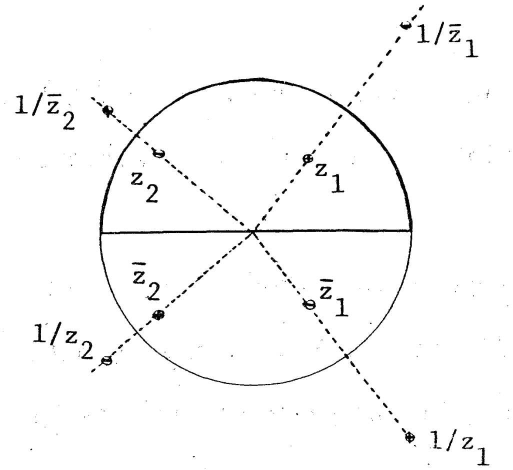


Fig. 1; The configuration of vortices

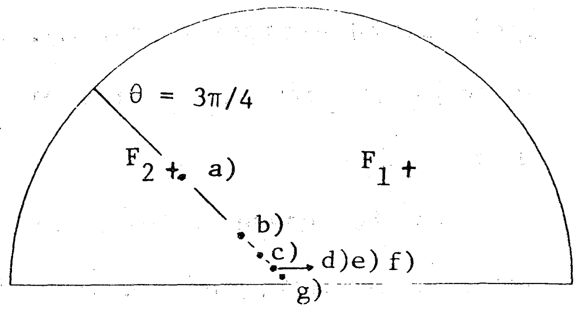
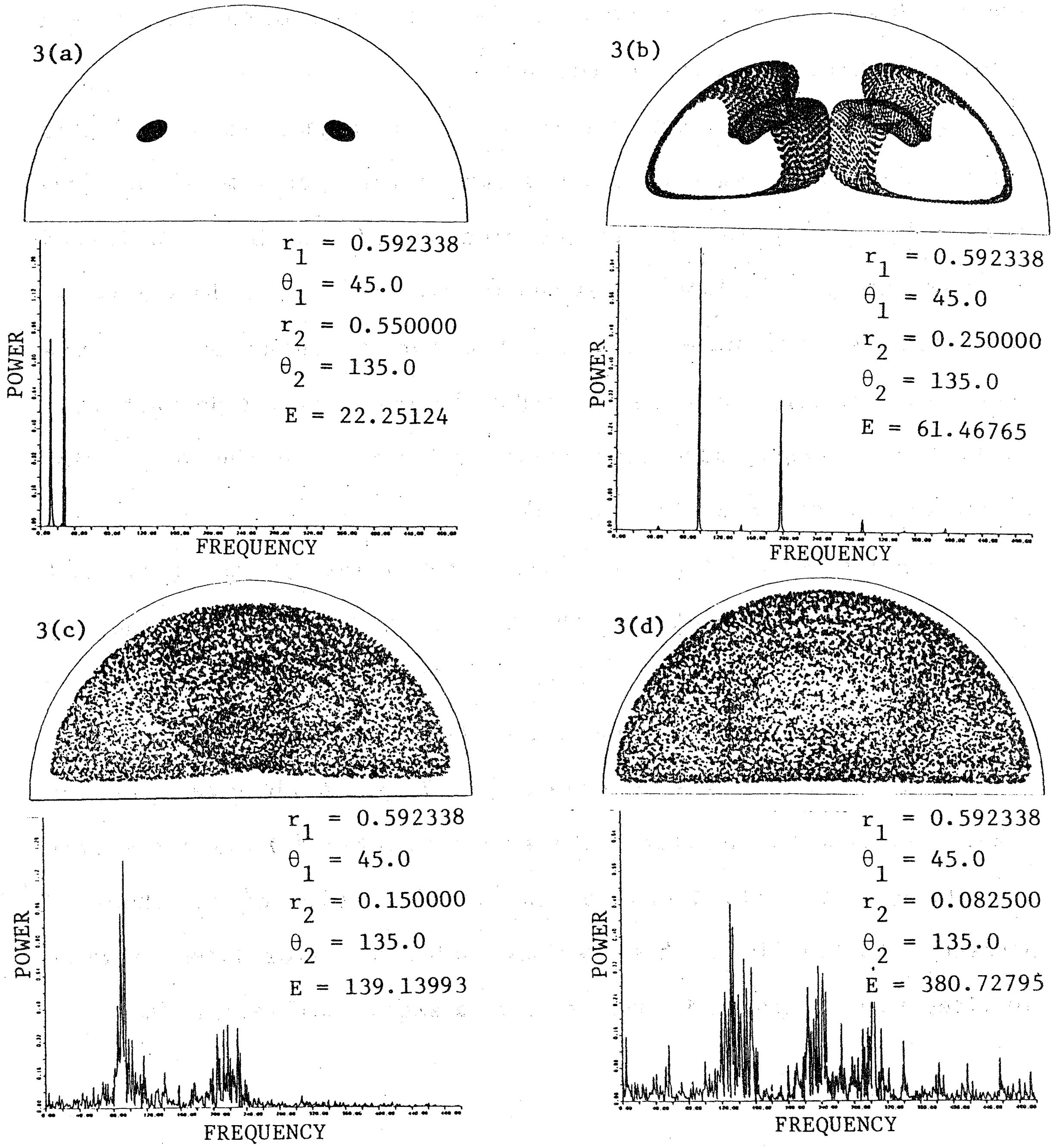
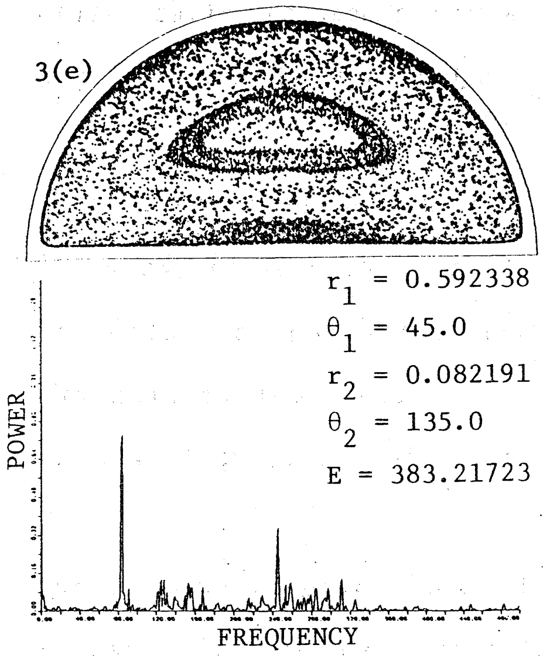
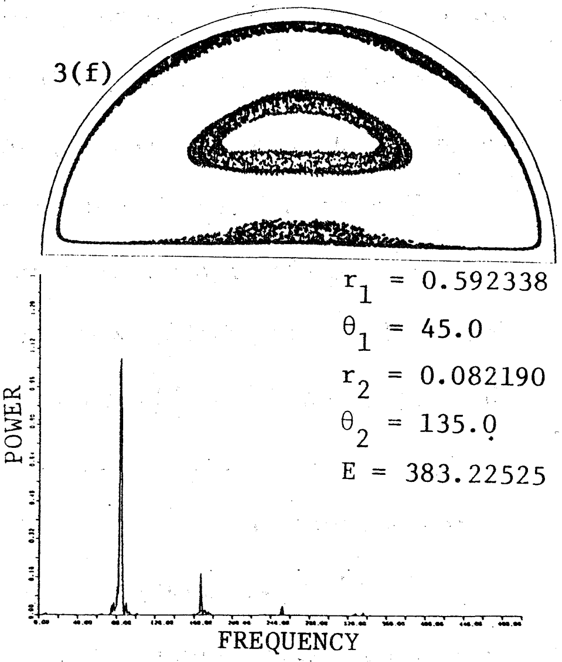


Fig. 2; The change of initial conditi







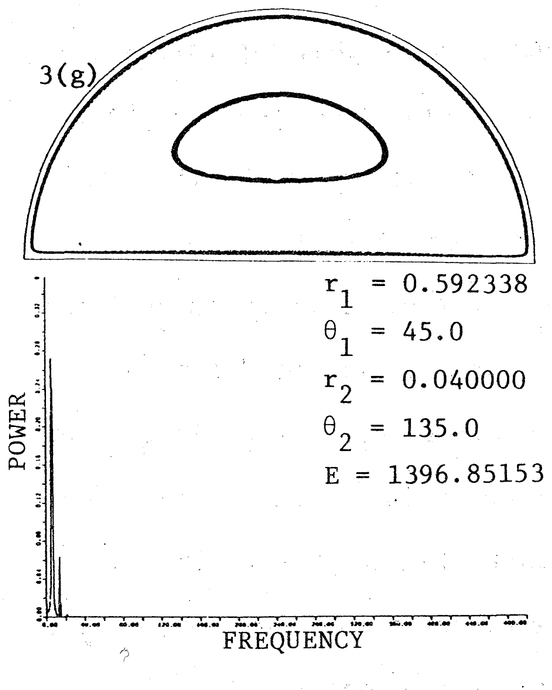


Fig. 3; Trajectories of two vortices and the power spectrum of r_1

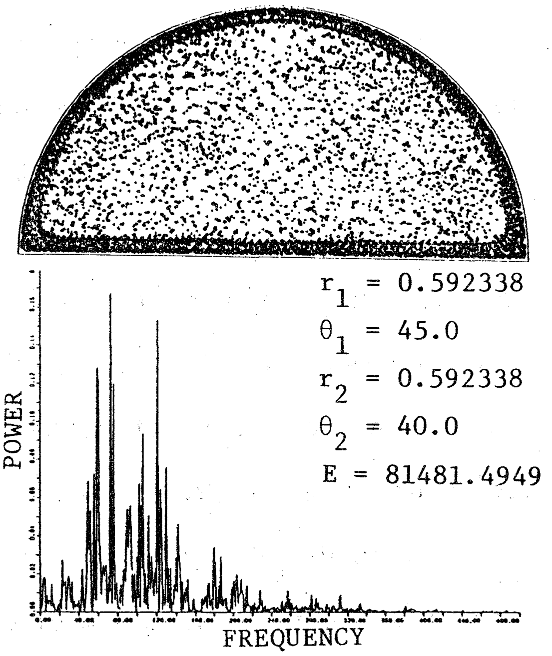


Fig. 4; Billiard type chaos

vortex makes a close vortex pair with its mirror image, and gives little effect on the motion of inner one whose trajectry is that of a single vortex in the semicircle.[11] (Fig.3(g))

The above-mentioned transition from regular state to another regular one through chaotic one is specified by means of the value of energy $E = \exp(-8\pi H)$. When $E \lesssim 114$, the former regular motion is observed, and $E \gtrsim 383$ corresponds to the latter. Both states are symmetric with respect to the y-axis. When $114 \lesssim E \lesssim 383$, irregular motion is obtained,

Finally we comment a different kind of motion from that are mentioned above. When one vortex is put near the other, they make a vortex pair and go straight to the boundary, where they separate and go along the boundary in different directions until they meet and form a vortex pair again. The repetition of this motion fills the whole semicircle, and affords a billiard type problem. (Fig. 4)

With the problem of the vortex chaos in a circular cylinder, Aref has presented a case of three vortices[13], and Hardin and Mason, a case of four vortices.[14] Our result is meaningful because we realize chaos with a lower freedom. We use a semicircular region as a prototype, but we can expect the same result with other shapes of regions like a rectangule, etc. As the first evidence of chaos, we adopted in this paper the trajectory of vortices and the power spectrum of the radius position of one vortex.

More details of the vortex chaos for example, the case of two identical vortices in a semicircle will be argued somewhere by means of other tools like the Poincare map or the Lyapnov exponents that characterize the chaotic behaviors in other ways.

4. MOTION IN A FLOW

As we mentioned in the introduction, existence of an external flow also diminishes the symmetry of the system. As long as the flow is incompressible, however, a streamfunction can be introduced, and included in the Hamiltonian. Let us denote the streamfunction to (x,y), then two components of the velocity of the external flow are given by

$$v_x = \frac{\partial}{\partial y} \psi \qquad , \qquad v_y = -\frac{\partial}{\partial x} \psi \ . \tag{4.1}$$

Thus, the equations of motion of vortices are written as

$$\Gamma_j \dot{x}_j = \frac{\partial}{\partial y_j} H_v + \Gamma_j \frac{\partial}{\partial y_j} \psi_j \ , \quad \Gamma_j \dot{y}_j = -\frac{\partial}{\partial x_j} H_v - \Gamma_j \frac{\partial}{\partial x_j} \psi_j \ , \tag{4.2}$$

where

$$\psi_j = \psi(x_j, y_j) \ , \tag{4.3}$$

and H_v denotes eq.(2.2);the interaction Hamiltonian among vortices. Introducing a new Hamiltonian

$$H = H_v + H_\psi \ , \tag{4.4}$$

where

$$H_\psi = \sum_{j=1}^{N} \Gamma_j \ \psi_j \ , \tag{4.5}$$

we obtain the same equation of motion as eq.(2.4). In the next subsection, we will mention the case of rotating uniform shear flow as an example.

<u>4-1. Rotating Uniform Shear Flow</u> Hereafter we use the complex variables z_j and $\bar{z}_j$ instead of x_j and y_j. If ψ is given as a quadratic form in z and $\bar{z}$;

$$\psi = \frac{1}{4}\left(A z^2 + \bar{A}\,\bar{z}^2 - 2\,\Omega\, z\,\bar{z} \right) \tag{4.6}$$

where A is the pure shear rate and is the angular velocity of the rotational motion, then the streamline is hyperbolic, parallel straight lines, and ellipse according to $|A| > \Omega$, $|A| = \Omega$ and $|A| < \Omega$, respectively.

We obtain the equation of motion of the center of vorticity by summing up eq.(4.2) and (4.3) with respect to j as

$$\frac{dG_z}{dt} = i\left(A\, G_z - \Omega\, \overline{G_z}\right) \tag{4.7}$$

where

$$G_z = \sum_{j=1}^{N} \Gamma_j\, z_j \tag{4.8}$$

Making use of eq.(4.7), the complex velocity w is given by

$$w = 2i\,\frac{\partial}{\partial z}\,\psi = i\left(A\, z - \Omega\, \bar{z}\right) \tag{4.9}$$

Thus, the orbit of G_z is perfectly in accordance with the streamline of the external flow.

Then we consider the relative motion of two vortices. Here we define a new variable $z_R = z_1 - z_2$. There are two cases according as the sum of two strengths is zero or not.

1) $\Gamma_1 + \Gamma_2 = 0$: vortex pair.

Dividing eq.(4.8) by Γ_1, we have

$$\dot{\bar{z}}_R = i(A\ z_R - \Omega\ z_R) \tag{4.10}$$

Thus the relative motion is governed by the same equation as eq.(4.8), i.e. the orbit in the z_R-plane should be hyperbola, parallel straight lines, and ellipse according to the same conditions with the case of the center of vorticity.

2) $\Gamma_1 + \Gamma_2 = \Gamma \neq 0$

Introducing the true center of vorticity

$$z_G = G_z / \Gamma \tag{4.11}$$

we can separate the Hamiltonian into two parts, i.e. H_G which depends only on z_G, and H_R only on z_R;

$$H = H_G + H_R \tag{4.12}$$

where

$$H_G = \Gamma\ \psi(z_G, \bar{z}_G) \tag{4.13}$$

and

$$H_R = \frac{\Gamma_1 \Gamma_2}{\Gamma} \left[\psi(z_R, \bar{z}_R) - \frac{\Gamma}{4\pi} \log z_R \bar{z}_R \right] \tag{4.14}$$

Thus the motion of z_G is obtainable as the trajectry of H_G = const., while z_R moves on the line of H_R = const., i.e.

$$R^2 = z_R \bar{z}_R = R_0^2 \exp\left[\frac{\pi}{\Gamma} (A\ z_R^2 - 2\ \Omega\ z_R \bar{z}_R + \bar{A}\ \bar{z}_R^2) \right]$$

$$R_0^2 = \exp\left[-4\pi H_R / \Gamma_1 \Gamma_2 \right] \tag{4.15}$$

This motion is classified by the three parameters A , Ω , and Γ .

5. DISCUSSION

We have so far investigated the motion of vortices in a boundary or an external flow from the viewpoint of a dynamical system. As a result, the existence of a boundary or an external flow breaks the symmetry of the system, which leads to the reduction of the number of the first integrals.

One important and subtle problem still remains in this subject. That is about an existence or a nonexistence of hidden integrals accoring to hidden symmetries of the system. With this problem, Ziglin[15], and Koiler and Calbalho[16] have proven the non-integrability of four point vortices in an unbounded region. Our results are entirely consistent with theirs, and we can draw conclusions about the nonexistence of the hidden integrals.

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This paper is dedicated to Professor Hidenori Hasimoto in celebration of his sixtieth birthday.

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